N23b Easter 2008

NATURAL SCIENCES TRIPOS
PART IB

Prof. M. E. McIntyre
mem@damtp.cam.ac.uk

Mathematical Methods III: Small Oscillations and Group Theory
Examples 2

1. Let Σ_3 be the permutation group for 3 objects. Show that $|\Sigma_3| = 6$, and that Σ_3 is isomorphic to D_3. By considering the action of Σ_3 in permuting the components of a vector $(a, b, c)^\mathsf{T}$ in 3-dimensional Euclidean space $\mathbb{R}^3$, or otherwise, show that the 3×3 unit matrix together with

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

provides a 3-dimensional faithful representation of Σ_3, where the isomorphism is defined by mapping the cycles (23), (31), (12), (123) and (132) to the matrices in the order shown.

Consider the matrix

$$S = \begin{pmatrix} \alpha & 0 & 2\beta \\ \alpha & \sqrt{3}\beta & -\beta \\ \alpha & -\sqrt{3}\beta & -\beta \end{pmatrix}$$

and show that it has determinant $|S| = -6\sqrt{3}\,\alpha\beta^2$. Show that, for any nonzero α and β, the matrix products $S^{-1}\{\text{matrices in first display above}\}S$ are equal to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

respectively. [Hint: first premultiply by S, and use the nonvanishing of $|S|$.]

How are these transformed matrices related to the 2-dimensional faithful representation of D_3 derived as a worked example in the lecture notes? Show that any transformation of this form, i.e., pre- and post-multiplication by a nonsingular matrix and its inverse, always defines an isomorphism. [Show that it will always turn a matrix group into another such group with the same multiplication table.]

2. Show that the symmetries of a tetrahedron in 3-dimensional space, including reflections (mirror images), form a group isomorphic to the permutation group Σ_4. Show that the same *without* reflections, i.e., the rigid rotations of a tetrahedron, is isomorphic to the alternating group A_4, the subgroup of Σ_4 consisting of even permutations only.
[A solution can be found on page 52 of the lecture notes.]

3. Use the permutation $\begin{pmatrix} 1\,2\,3\,4\,5 \\ a\;b\;c\;d\;e \end{pmatrix}$ to show that two permutations with (disjoint) cycle decompositions $(123)(45)$ and $(abc)(de)$ are in the same conjugacy class within Σ_5. Generalize this example to show that two non-indentity permutations are in the same conjugacy class, within Σ_5, if and only if their cycle decompositions have the same cycle shape $(\cdot\ \cdot)$, $(\cdot\ \cdot)(\cdot\ \cdot)$, $(\cdot\ \cdot\ \cdot)$, $(\cdot\ \cdot\ \cdot)(\cdot\ \cdot)$, etc. Deduce that there are 7 conjugacy classes in Σ_5.

4. Consider the following mappings from D_4 into or onto C_2, with C_2 represented as $\{1, -1\}$:

$$\{I, R, R^2, R^3, m_1, m_2, m_3, m_4\} \mapsto \{1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1\}$$
$$\{I, R, R^2, R^3, m_1, m_2, m_3, m_4\} \mapsto \{1,\ 1,\ 1,\ 1,\ -1,\ -1,\ -1,\ -1\}$$
$$\{I, R, R^2, R^3, m_1, m_2, m_3, m_4\} \mapsto \{1,\ -1,\ 1,\ -1,\ 1,\ 1,\ -1,\ -1\}$$
$$\{I, R, R^2, R^3, m_1, m_2, m_3, m_4\} \mapsto \{1,\ -1,\ 1,\ -1,\ -1,\ -1,\ 1,\ 1\}$$
$$\{I, R, R^2, R^3, m_1, m_2, m_3, m_4\} \mapsto \{1,\ -1,\ -1,\ 1,\ 1,\ -1,\ -1,\ 1\}$$

in the order displayed. Show that the first four are homomorphisms but that the last is not. Verify that the kernels of the first four mappings are all normal subgroups of D_4, and that the kernel of the last mapping is not.

5. Show that $Tr(AB) = Tr(BA)$. Deduce that $Tr(S^{-1}DS) = Tr\,D$.
Let R_1 denote the 3×3 rotation matrix for a rotation by π about the direction $(1, 0, 0)$, and R_2 the matrix for a rotation by π about the direction $\frac{1}{\sqrt{2}}(1, 1, 0)$. Verify that $Tr\,R_1 = Tr\,R_2$. Find an invertible matrix S, such that $S^{-1}\,R_1\,S = R_2$.
[Hint: It is easier to solve $R_1\,S = SR_2$. Note that the answer is not unique.]

6. For a group G show that for any $g_1 \in G$ the elements $\{h\}$ such that $hg_1h^{-1} = g_1$ form a subgroup H_{g_1}. Show that if $gg_1g^{-1} = g_2$ for some $g \in G$, then H_{g_1} is isomorphic to H_{g_2}. Show that the conjugacy class of g_1 has $|G|/|H_{g_1}|$ elements.

7. Let G be an abelian group with $|G|$ elements. Show that each element of G forms a conjugacy class by itself. Deduce that there are $|G|$ one-dimensional representations of G and no other irreducible representations. Find the one-dimensional representations of the cyclic group C_n.

8. Let $\mathbf{e}_1, \mathbf{e}_2$ be unit vectors in the plane separated by an angle of $120°$, Δ the equilateral triangle with vertices $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3 = -(\mathbf{e}_1 + \mathbf{e}_2)$ and Σ_3 the symmetry group of Δ.
Calculate the matrices of the two-dimensional irreducible representation of Σ_3 by considering the action on vectors in the plane, taking $\mathbf{e}_1$ and $\mathbf{e}_2$ as basis vectors. Show that the traces of these matrices agree with those in the character table of Σ_3. Verify that the orthogonality theorem

$$\sum_g d^{(\alpha)}(g)_{ij} d^{(\alpha)}(g^{-1})_{kl} = |G|/n_\alpha \delta_{il}\delta_{jk}$$

is satisfied in this case, as it must be.

9. Let D be a unitary representation of a finite group G and $\{\chi(g) : g \epsilon G\}$ the character of D. Show that

$$\frac{1}{|G|} \sum_g \chi(g)^* \chi(g)$$

is a positive integer, equal to 1 if and only if D is irreducible.