1. Let $\Sigma_3$ be the permutation group for 3 objects. Show that $|\Sigma_3| = 6$, and that $\Sigma_3$ is isomorphic to $D_3$. By considering the action of $\Sigma_3$ in permuting the components of a vector $(a, b, c)^T$ in 3-dimensional Euclidean space $\mathbb{R}^3$, or otherwise, show that the $3 \times 3$ unit matrix together with
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}
\]
provides a 3-dimensional faithful representation of $\Sigma_3$, where the isomorphism is defined by mapping the cycles $(23), (31), (12), (123)$ and $(132)$ to the matrices in the order shown.

Consider the matrix
\[
S = \begin{pmatrix}
\alpha & 0 & 2\beta \\
\alpha & \sqrt{3}\beta & -\beta \\
\alpha & -\sqrt{3}\beta & -\beta
\end{pmatrix}
\]
and show that it has determinant $|S| = -6\sqrt{3}\alpha\beta^2$. Show that, for any nonzero $\alpha$ and $\beta$, the matrix products $S^{-1}\{\text{matrices in first display above}\}S$ are equal to
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\
0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2}
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2}
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 \\
0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\
0 & \frac{\sqrt{3}}{2} & -\frac{1}{2}
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 \\
0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\
0 & \frac{\sqrt{3}}{2} & -\frac{1}{2}
\end{pmatrix}
\]
respectively. [Hint: first premultiply by $S$, and use the nonvanishing of $|S|$.

How are these transformed matrices related to the 2-dimensional faithful representation of $D_3$ derived as a worked example in the lecture notes? Show that any transformation of this form, i.e., pre- and post-multiplication by a nonsingular matrix and its inverse, always defines an isomorphism. [Show that it will always turn a matrix group into another such group with the same multiplication table.]

2. Show that the symmetries of a tetrahedron in 3-dimensional space, including reflections (mirror images), form a group isomorphic to the permutation group $\Sigma_4$. Show that the same without reflections, i.e., the rigid rotations of a tetrahedron, is isomorphic to the alternating group $A_4$, the subgroup of $\Sigma_4$ consisting of even permutations only.

[A solution can be found on page 52 of the lecture notes.]

3. Use the permutation $(1 \ 2 \ 3 \ 4 \ 5 \ a \ b \ c \ d \ e)$ to show that two permutations with (disjoint) cycle decompositions $(123)(45)$ and $(abc)(de)$ are in the same conjugacy class within $\Sigma_5$. Generalize this example to show that two non-identity permutations are in the same conjugacy class.
class, within $\Sigma_5$, if and only if their cycle decompositions have the same cycle shape $(\cdot \cdot)$, $(\cdot \cdot)(\cdot \cdot)$, $(\cdot \cdot \cdot)(\cdot \cdot)$, etc. Deduce that there are 7 conjugacy classes in $\Sigma_5$.

4. Consider the following mappings from $D_4$ into or onto $C_2$, with $C_2$ represented as $\{1, -1\}$:

\[
\begin{align*}
\{I, R, R^2, R^3, m_1, m_2, m_3, m_4\} &\mapsto \{1, 1, 1, 1, 1, 1, 1, 1\} \\
\{I, R, R^2, R^3, m_1, m_2, m_3, m_4\} &\mapsto \{1, 1, 1, -1, -1, -1, -1, -1\} \\
\{I, R, R^2, R^3, m_1, m_2, m_3, m_4\} &\mapsto \{-1, 1, -1, 1, 1, -1, 1, -1\} \\
\{I, R, R^2, R^3, m_1, m_2, m_3, m_4\} &\mapsto \{-1, 1, -1, -1, -1, 1, 1, 1\} \\
\{I, R, R^2, R^3, m_1, m_2, m_3, m_4\} &\mapsto \{1, -1, 1, -1, 1, -1, 1, 1\}
\end{align*}
\]

in the order displayed. Show that the first four are homomorphisms but that the last is not. Verify that the kernels of the first four mappings are all normal subgroups of $D_4$, and that the kernel of the last mapping is not.

5. Show that $Tr(AB) = Tr(BA)$. Deduc that $Tr(S^{-1}DS) = TrD$.

Let $R_1$ denote the $3 \times 3$ rotation matrix for a rotation by $\pi$ about the direction $(1,0,0)$, and $R_2$ the matrix for a rotation by $\pi$ about the direction $\frac{1}{\sqrt{2}} (1,1,0)$. Verify that $Tr R_1 = TrR_2$. Find an invertible matrix $S$, such that $S^{-1} R_1 S = R_2$.

[Hint: It is easier to solve $R_1 S = S R_2$. Note that the answer is not unique.]

6. For a group $G$ show that for any $g_1 \in G$ the elements $\{h\}$ such that $h g_1 h^{-1} = g_1$ form a subgroup $H_{g_1}$. Show that if $g_1 g_2^{-1} = g_2$ for some $g \in G$, then $H_{g_1}$ is isomorphic to $H_{g_2}$. Show that the conjugacy class of $g_1$ has $|G|/|H_{g_1}|$ elements.

7. Let $G$ be an abelian group with $|G|$ elements. Show that each element of $G$ forms a conjugacy class by itself. Deduce that there are $|G|$ one-dimensional representations of $G$ and no other irreducible representations. Find the one-dimensional representations of the cyclic group $C_n$.

8. Let $e_1, e_2$ be unit vectors in the plane separated by an angle of $120^\circ$, $\Delta$ the equilateral triangle with vertices $e_1, e_2$ and $e_3 = -(e_1 + e_2)$ and $\Sigma_3$ the symmetry group of $\Delta$.

Calculate the matrices of the two-dimensional irreducible representation of $\Sigma_3$ by considering the action on vectors in the plane, taking $e_1$ and $e_2$ as basis vectors. Show that the traces of these matrices agree with those in the character table of $\Sigma_3$. Verify that the orthogonality theorem

\[
\sum_g d^{(\alpha)}(g)_{ij} d^{(\alpha)(g^{-1})}_{kl} = |G| n_{\alpha} \delta_{ii} \delta_{jk}
\]

is satisfied in this case, as it must be.

9. Let $D$ be a unitary representation of a finite group $G$ and $\{\chi(g) : g \in G\}$ the character of $D$. Show that

\[
\frac{1}{|G|} \sum_g \chi(g)^* \chi(g)
\]

is a positive integer, equal to 1 if and only if $D$ is irreducible.