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A note on the divergence effect and the Lagrangian-mean surface elevation in periodic water waves

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Longuet-Higgins' exact expression for the increase in the Lagrangian-mean elevation of the free surface due to the presence of periodic, irrotational surface gravity waves is rederived from generalized Lagrangian-mean theory. The raising of the Lagrangian-mean surface as wave amplitude builds up illustrates the non-zero divergence of the Lagrangian-mean velocity field in an incompressible fluid.

1. Introduction

Wave-induced Lagrangian-mean velocities – defined, following Stokes (1847), Longuet-Higgins (1953), and others, as rates of change of mean particle positions – are generally divergent even in an incompressible fluid. This interesting fact was first pointed out to me by D. G. Andrews (McIntyre 1973, p. 810) and was further discussed in our joint papers on the generalized Lagrangian-mean theory and related matters (Andrews & McIntyre 1976, 1978*a*, hereinafter AM76, AM78 respectively) and in subsequent papers including those by Grimshaw (1979), Uryu (1979), McIntyre (1980), Dunkerton *et al.* (1981), and Nakamura (1981). The phenomenon is akin to the well-known divergence, or dispersion, of particle paths in incompressible turbulent fluid motion.

When asymptotic expansions are used to describe the waves and their mean effects, the order of approximation at which the divergence effect is significant depends upon the particular wave problem and upon which variables are considered to be of interest. For example the divergence effect is significant at leading order in the problem studied in McIntyre (1973), namely that of calculating correct to $O(a^2)$ the mean velocity field and impulse of a dissipationless, two-dimensional, horizontally guided internal gravity wave packet of small amplitude a. The problem is of theoretical interest as an exception to the usual rule for the mean force on an immersed obstacle scattering the waves (e.g. McIntyre & Mobbs 1988). For an incompressible fluid, it can be shown quite generally (for rotational as well as irrotational waves) that the Lagrangian-mean velocity field $\boldsymbol{\bar{u}}^{\mathrm{L}}(\boldsymbol{x}, t)$ satisfies

$$\boldsymbol{\nabla}\cdot\boldsymbol{\bar{u}}^{\mathrm{L}} = \frac{1}{2}\left(\frac{\partial}{\partial t}+\boldsymbol{\bar{u}}^{\mathrm{L}}\cdot\boldsymbol{\nabla}\right)\frac{\partial^2}{\partial x_i\,\partial x_j}\overline{(\xi_i\,\xi_j)} \tag{1}$$

correct to $O(a^2)$ (AM78, equation 9.4). Here the summation convention is used, $\xi_i(\boldsymbol{x}, t)$ is the ith component of the material particle displacement $\boldsymbol{\xi}(\boldsymbol{x}, t)$ about position $\boldsymbol{x}$ (ith component x_i), defined correct to $O(a)$, and $\overline{(\;)}$ is a suitable Eulerian averaging operator. The right-hand side of (1) generally differs from zero, significantly so in the problem just mentioned. Some supporting details are given in the Appendix.

Another example which has been studied in detail, and in which the divergence effect is significant at leading order, is the problem of calculating the Lagrangian-mean flow induced by a growing Eady baroclinic-instability wave (Uryu 1979). If the right-hand side of (1) were taken to be zero, a calculation of $\bar{\boldsymbol{u}}^{\mathrm{L}}(\boldsymbol{x}, t)$ from the Lagrangian-mean equations of motion would give a qualitatively incorrect answer. Yet another such example is the mean flow induced by equatorially trapped waves on a rotating planet. These waves are observed in the Earth's equatorial lower stratosphere, where they are believed on strong evidence to play an important role in bringing about the observed reversals, every thirteen months or so, of the east–west mean winds that encircle the globe in that region. For details, and references, the reader may consult AM76, and AM78, §9.

The divergence effect can also be significant, for certain purposes, in classical problems of irrotational surface gravity waves. The simplest example is the generation of irrotational, periodic, small-amplitude waves from rest by surface pressure fluctuations. Equation (1) reduces to

$$\frac{\partial \bar{w}^{\mathrm{L}}}{\partial z} = \frac{1}{2}\frac{\partial^3}{\partial t\,\partial z^2}\overline{(\zeta^2)}, \tag{2}$$

where z $(= x_3)$, w $(= u_3)$ and ζ $(= \xi_3)$ are directed vertically upwards. The Eulerian averaging operator $\overline{(\;)}$ can be taken as a horizontal average in $x(= x_1)$. The left-hand side of (2) can be written as $\partial^2 Z/\partial t\,\partial z$, where $Z(z, t)$ denotes the $O(a^2)$ mean vertical displacement of a material particle from its undisturbed position z. Integrating forwards in time from an initial state of no disturbance, and upwards from undisturbed deep water, or from a flat lower boundary on which $\partial\overline{(\zeta^2)}/\partial z = 0$, we have

$$Z(z, t) = \frac{1}{2}\frac{\partial}{\partial z}\overline{(\zeta^2)}. \tag{3}$$

We note that ρg times the z-integral of this expression over the full depth, where ρ is the density and g the gravity acceleration (both constant), agrees with the usual expression for the increase in potential energy per unit area due to the presence of surface gravity waves. That expression is $\frac{1}{2}\rho g\overline{\zeta^2}$, evaluated at the free surface. This gives an independent check on the correctness of (1), since one way of evaluating the potential energy is evidently in terms of the mass-weighted mean elevation $Z(z, t)$ of all fluid particles relative to their undisturbed positions. For internal gravity waves under a rigid upper lid we may multiply (3) by $\rho g = \rho(z)\,g$ and integrate by parts to get the z-integral of $\frac{1}{2}g(-\partial\rho/\partial z)\,\overline{\zeta^2}$. This verifies the standard $O(a^2)$ form of the wave-induced potential energy for the internal-wave problem.†

For the case of a progressive, deep-water surface gravity wave $\zeta = a \exp kz \cos k(x-ct)$, we have $\overline{\zeta^2} = \frac{1}{2}a^2 \exp 2kz$. Then (3) shows that the value Z_{s} of Z at the free surface $z = 0$ is given, correct to $O(a^2)$, by

$$Z_{\mathrm{s}} = \tfrac{1}{2}a^2 k = \frac{\frac{1}{2}\bar{u}_0^{\mathrm{L}}\, c}{g}, \tag{4a, b}$$

where $\bar{u}_0^{\mathrm{L}}$ denotes the Lagrangian-mean horizontal velocity component evaluated at

† A third independent check is given by comparing (1) with (A 2) of McIntyre (1973), and a fourth by the calculations described in AM78, equations (9.9) ff; see also Grimshaw (1979), equation (4.10) ff.

the free surface, in a frame of reference fixed relative to deep water. The second expression ($4b$) follows from the first, ($4a$), on setting $z = 0$ in the well-known $O(a^2)$ expression $\bar{u}^{\mathrm{L}}(z) = a^2k^2c \exp 2kz$ (Stokes 1847, equation 23), and using the deep-water dispersion relation $c^2 = g/k$.

In his invited paper to the G. I. Taylor Symposium volume of this journal, Longuet-Higgins (1986, hereinafter LH) derives the two expressions ($4a, b$) by a different route, independent of (1), and suggests that they may have practical implications for certain geophysical measurements involving instrumented buoys on the sea surface (see also Srokosz & Longuet-Higgins 1986, who derive an irrotational version of (3), and Longuet-Higgins 1987, 1988). LH shows further that, surprisingly, the second expression ($4b$) holds exactly, for waves of finite amplitude. It depends only upon assuming periodic waves, irrotational motion, and deep water. In the remainder of this note it is shown how the exact result ($4b$) can be deduced from the generalized Lagrangian-mean (GLM) theory from which (1) was originally derived. Section 2 recalls some basic GLM definitions and relations, and then gives a succinct derivation of ($4b$) the essential idea for which was suggested to me by D. G. Andrews (personal communication). Section 3 gives an alternative derivation which is less direct but which allows one to see the relationship between ($4b$) and certain exact, general theorems that apply to rotational as well as to irrotational waves in fluids.

2. A direct derivation of the exact formula (4*b*) from GLM theory

The Lagrangian-mean velocity $\bar{\boldsymbol{u}}^{\mathrm{L}}$, for a general velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ and a general Eulerian averaging operator $\overline{(\ )}$, is defined by

$$\overline{\boldsymbol{u}(\boldsymbol{x}, t)}^{\mathrm{L}} = \overline{\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{\xi}(\boldsymbol{x}, t), t)}. \tag{5}$$

Here $\boldsymbol{\xi}(\boldsymbol{x}, t)$ is the displacement about mean position $\boldsymbol{x}$, i.e. defined such that

$$\overline{\boldsymbol{\xi}(\boldsymbol{x}, t)} = 0. \tag{6}$$

These definitions are exact; a careful discussion of their basis, for arbitrary space, time or ensemble averaging operators $\overline{(\ )}$, is given in AM78. In the present problem $\overline{(\ )}$ can be taken, as before, to be a spatial average with respect to horizontal distance x. The GLM definitions are fully consistent with, but also represent a significant generalization of, the classical definitions of Stokes (1847, equations (22) ff.) and Longuet-Higgins (1953, equation (21)); see the Appendix below and, for further discussion, §§1, 2.1 of McIntyre (1980). For instance the GLM formalism overcomes the problem, noted by Longuet-Higgins (1953, p. 541), of defining $\bar{\boldsymbol{u}}^{\mathrm{L}}$ to higher accuracy than $O(a^2)$.

The generalized Lagrangian mean of an arbitrary field $\psi(\boldsymbol{x}, t)$ is defined in the same way as in (5); that is, one replaces $\boldsymbol{x}$ by $\boldsymbol{x}+\boldsymbol{\xi}(\boldsymbol{x}, t)$ and then applies the Eulerian averaging operator $\overline{(\ )}$. The operator $\overline{(\ )}^{\mathrm{L}}$, thus defined, is useful because it gives a simple result when applied to the material derivative, namely

$$\overline{\left(\frac{\mathrm{D}\psi}{\mathrm{D}t}\right)}^{\mathrm{L}} = \overline{\left(\frac{\partial\psi}{\partial t}+\boldsymbol{u}\cdot\boldsymbol{\nabla}\psi\right)}^{\mathrm{L}} = \frac{\partial\bar{\psi}^{\mathrm{L}}}{\partial t}+\bar{\boldsymbol{u}}^{\mathrm{L}}\cdot\boldsymbol{\nabla}\bar{\psi}^{\mathrm{L}} \tag{7}$$

(AM78, equation (2.15)).

In order to derive ($4b$) it is convenient to go into a frame of reference following the waves, as in LH, so that the flow is steady. The Lagrangian-mean velocity $\bar{\boldsymbol{u}}^{\mathrm{L}}$ now approaches $(-c, 0)$ as $z \to -\infty$. Its horizontal component

$$\bar{u}^{\mathrm{L}}(z) = \bar{u}_0^{\mathrm{L}}(z) - c, \tag{8}$$

where $\bar{u}_0^{\mathrm{L}}$ again refers to the frame of reference in which the fluid at depth is motionless, but is now defined for general z.

Equation (4*b*) is the result of applying the operator $\overline{(\ )}^{\mathrm{L}}$ to the steady-state Bernoulli equation. With a suitable choice of origin for z, to be specified in a moment, the Bernoulli equation can be taken as

$$\tfrac{1}{2}|\boldsymbol{u}|^2 + \frac{p}{\rho} + gz = \tfrac{1}{2}c^2. \tag{9}$$

Since at great depth the pressure is horizontally uniform and steady (in contrast with problems involving standing waves), and depends hydrostatically on the mass of the overlying layer whether or not the waves are present (Lamb 1932, p. 420), (9) is the appropriate form when the z-origin is taken in the undisturbed free surface. Introducing the velocity potential

$$\phi(\boldsymbol{x}, t) = -cx + \phi_0(\boldsymbol{x}, t), \tag{10}$$

say, we note that

$$|\boldsymbol{u}|^2 = \boldsymbol{u}\cdot\nabla\phi = -cu + \boldsymbol{u}\cdot\nabla\phi_0, \tag{11}$$

and that ϕ_0 is periodic in x so that its Lagrangian mean exists and is x-independent. Now since $\partial/\partial t = 0$, (11) and (7) give immediately

$$\overline{|\boldsymbol{u}|^2}^{\mathrm{L}} = -c\bar{u}^{\mathrm{L}} + \bar{\boldsymbol{u}}^{\mathrm{L}}\cdot\nabla\bar{\phi}_0^{\mathrm{L}} = -c\bar{u}^{\mathrm{L}}. \tag{12}$$

The last step exploits the fact that $\bar{w}^{\mathrm{L}}$ is exactly zero in the steady state, as well as the fact that $\partial\bar{\phi}_0^{\mathrm{L}}/\partial x = 0$. The vanishing of $\bar{w}^{\mathrm{L}}$ for steady, irrotational, periodic water waves is evident from the definition (5) and the spatial symmetry of the velocity and displacement fields in such waves. Figure 1 depicts the displacements of material particles lying on a material contour whose undisturbed position is horizontal; note the symmetry of the displacement field about each wavecrest, ζ being an even function and ξ an odd function of x.

Applying the operator $\overline{(\ )}^{\mathrm{L}}$ to (9), using (12), and noting that (6) implies that gz equals its Lagrangian mean, we get

$$-\tfrac{1}{2}c\bar{u}^{\mathrm{L}} + \frac{\bar{p}^{\mathrm{L}}}{\rho} + gz = \tfrac{1}{2}c^2. \tag{13}$$

Rearranging this result and using (8), we see that

$$gz = \tfrac{1}{2}c^2 - \frac{\bar{p}^{\mathrm{L}}}{\rho} + \tfrac{1}{2}\bar{u}^{\mathrm{L}}c = \tfrac{1}{2}\bar{u}_0^{\mathrm{L}}c - \frac{\bar{p}^{\mathrm{L}}}{\rho}. \tag{14}$$

Since z is by definition the exact Lagrangian-mean position, relative to the undisturbed free surface, of the associated material contour (again see (6) and figure 1), the relation (14) gives the exact Lagrangian-mean depth of the material contour for which the Lagrangian-mean pressure is $\bar{p}^{\mathrm{L}}$. In particular, continuing analytically to the free surface (where p and therefore $\bar{p}^{\mathrm{L}}$ vanish after wave generation), we see that (14) reduces exactly to Longuet-Higgins' result (4*b*).

FIGURE 1. Sketch of an initially horizontal material contour and the associated displacement field $\boldsymbol{\xi}(\boldsymbol{x}, t)$ in a deep-water wave, for one value of z and a number of equally spaced values of x. The horizontal line marks the given value of z. By (6), the latter gives the mean depth of the set of material particles located at the tips of the displacement arrows (i.e. at positions $\boldsymbol{x}+\boldsymbol{\xi}$). Note that, because the average is evenly weighted over x rather than $x+\xi$, the horizontal line lies higher than a line cutting off equal areas.

3. An alternative GLM derivation

For an inviscid, incompressible, homogeneous fluid of constant density ρ in an inertial frame of reference, equation (3.13) (Theorem II) of AM78 reduces to the exact relation

$$\left(\frac{\partial}{\partial t}+\overline{\boldsymbol{u}}^{\mathrm{L}}\cdot\boldsymbol{\nabla}\right)\left(\tfrac{1}{2}\overline{|\boldsymbol{u}|^2}^{\mathrm{L}}+\frac{\overline{p}^{\mathrm{L}}}{\rho}+gz-\boldsymbol{e}\right)=\frac{\partial\overline{\boldsymbol{u}}^{\mathrm{L}}}{\partial t}\cdot(\overline{\boldsymbol{u}}^{\mathrm{L}}-\boldsymbol{q})+\frac{\partial}{\partial t}\left(-\tfrac{1}{2}\overline{|\boldsymbol{u}|^2}^{\mathrm{L}}+\frac{\overline{p}^{\mathrm{L}}}{\rho}+gz\right), \quad (15)$$

where $\boldsymbol{e}$ is the pseudoenergy, or quasi-energy, associated with the wave motion, and $\boldsymbol{q}$ the corresponding pseudomomentum or quasi-momentum.† These are defined (in the inertial frame) as

$$\boldsymbol{e}=\overline{\left(\frac{\partial\boldsymbol{\xi}}{\partial t}\right)\cdot\boldsymbol{u}'}, \quad (16)$$

$$\boldsymbol{q}=-\overline{(\boldsymbol{\nabla}\boldsymbol{\xi})\cdot\boldsymbol{u}'}, \quad (17)$$

the contraction in the last expression being with respect to the components of $\boldsymbol{\xi}$ and not $\boldsymbol{\nabla}$. Here $\boldsymbol{u}'$ denotes the Lagrangian disturbance velocity field, defined as $\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{\xi}(\boldsymbol{x},t),t)-\overline{\boldsymbol{u}}^{\mathrm{L}}(\boldsymbol{x},t)$. For irrotational motion we have, in addition, the exact relation

$$\boldsymbol{\nabla}\times(\overline{\boldsymbol{u}}^{\mathrm{L}}-\boldsymbol{q})=0 \quad (18)$$

(AM78 theorem I, corollary IV). The reader interested in verifying these results is referred to AM78. (Theorems I and II are respectively the spatial, and minus the temporal, components of a set of four equations obtained by contracting the 4-tensor $\boldsymbol{\nabla}_{(4)}(\boldsymbol{\Xi}, t)$ with the equations of motion evaluated at position $\boldsymbol{\Xi}(\boldsymbol{x},t)=\boldsymbol{x}+\boldsymbol{\xi}(\boldsymbol{x},t)$ and time t, and then applying the Eulerian averaging operator $\overline{(\;)}$. Here $\boldsymbol{\nabla}_{(4)}$ is the four-dimensional gradient operator $(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3, \partial/\partial t)$, 'equations of motion'

† The prefix 'quasi-', of whose currency Andrews and I were unaware in 1978, seems etymologically the more apt. In GLM theory the quasi-energy and quasi-momentum are the conservable, $O(a^2)$ wave properties associated respectively with temporal and spatial invariance of the Lagrangian-mean flow. Many types of waves behave, for many purposes, *as if* they had energy $\boldsymbol{e}$ and momentum $\boldsymbol{q}$ (Andrews & McIntyre 1978*a*, *b*; McIntyre & Mobbs 1988), the internal-wave problem studied in McIntyre (1973) being an exception to this rule. By $O(a^2)$ wave properties is meant quantities evaluable correct to $O(a^2)$ from linear theory, to $O(a^3)$ from second-order theory, and so on to higher orders in a. Analogous $O(a^2)$ wave properties may also be defined, albeit in a somewhat less general way, in terms of Eulerian variables (McIntyre & Shepherd 1987, §7, and references therein), following Hamiltonian concepts developed by Poisson, Lie, Dirac, Arnol'd and others. We note that AM78's Theorem II has been further discussed by Dunkerton (1983), who also illustrates the kinematics involved in defining $\overline{(\;)}^{\mathrm{L}}$ when $\overline{(\;)}$ is a time average.

means the momentum equations together with the negative† of the energy equation, per unit mass, and the contraction is taken with respect to the components of (Ξ, t), not $\nabla_{(4)}$, as in (16), (17) above.)

Now we are at liberty to imagine that the waves are generated from rest, in any way we please. In particular, we may imagine that they are generated as slowly as we please, by an arbitrarily weak, stationary, periodic pressure pattern applied at the surface. Let the magnitude of the imposed pressure be characterized by the small parameter μ, and let the pressure be applied for a time of order μ^{-1}. Then $\partial/\partial t$, $\boldsymbol{e}$, and the vertical components of $\overline{\boldsymbol{u}}^{\mathrm{L}}$ and $\boldsymbol{q}$ are all $O(\mu)$ during wave generation and all exactly zero in the final steady state. The vanishing of the vertical component of $\boldsymbol{q}$, for steady, irrotational, periodic waves, is evident from the spatial symmetry properties of the velocity and displacement fields in such waves already referred to below equation (12).

The Bernoulli function $\frac{1}{2}|\boldsymbol{u}|^2 + (p/\rho) + gz$ is now equal to

$$\tfrac{1}{2}c^2 - \frac{\partial \phi}{\partial t} = \tfrac{1}{2}c^2 + O(\mu),$$

in the circumstances assumed. It follows that the left-hand side of (15) is $O(\mu^2)$. The $\partial \overline{w}^{\mathrm{L}}/\partial t$ contribution to the first term on the right of (15) is $O(\mu^3)$. Therefore

$$\frac{\partial \overline{u}^{\mathrm{L}}}{\partial t}(\overline{u}^{\mathrm{L}} - q) + \frac{\partial}{\partial t}\left(-\tfrac{1}{2}\overline{|\boldsymbol{u}|^2}^{\mathrm{L}} + \frac{\overline{p}^{\mathrm{L}}}{\rho} + gz\right) = O(\mu^2), \tag{19}$$

where q is the horizontal component of $\boldsymbol{q}$. But (18) implies that $(\overline{u}^{\mathrm{L}} - q)$ is independent of depth z, and so must be equal to its value at $z = -\infty$, which value is $-c$. Therefore the left-hand side of (19) can be integrated over the time interval of order μ^{-1} during which the waves are generated to give

$$-\overline{u}^{\mathrm{L}}c - \tfrac{1}{2}\overline{|\boldsymbol{u}|^2}^{\mathrm{L}} + \frac{\overline{p}^{\mathrm{L}}}{\rho} + gz = \mathrm{func}\,(z) + O(\mu). \tag{20}$$

Now in the initial, undisturbed, hydrostatic state, the left-hand side of (20) is equal to a constant. The $O(\mu)$ contribution on the right can be made as small as we please, relative to the left-hand side, by generating the waves arbitrarily slowly. If, therefore, the origin is now taken as before in the undisturbed free surface, the function of z on the right takes the constant value $\frac{1}{2}c^2$, and it follows that

$$-\overline{u}^{\mathrm{L}}c - \tfrac{1}{2}\overline{|\boldsymbol{u}|^2}^{\mathrm{L}} + \frac{\overline{p}^{\mathrm{L}}}{\rho} + gz = \tfrac{1}{2}c^2 \tag{21}$$

in the final steady, periodic wave motion. Adding half of (21) to half the Lagrangian mean of (9) immediately recovers (13), and hence (14) and (4*b*).

I am grateful to D. G. Andrews and two anonymous referees for constructive comments on the first draft of this note, which led to a number of improvements. In particular, Andrews spotted the elegant gambit of rewriting $|\boldsymbol{u}|^2$ in the manner shown in (11), which led to the very succinct derivation of (4*b*) given in the second half of §2.

† The minus signs appear because it is the energy–momentum one-form that is involved in the contraction, not the energy–momentum vector.

Appendix. Equation (1) when $|\overline{\boldsymbol{u}}^{\mathrm{L}}| = O(a^2)$

We rederive (1) in the circumstances envisaged in §1, in which classical definitions of the displacement and Lagrangian-mean velocity fields apply. It is shown in particular that there is no conflict with Longuet-Higgins' elegant proof (1953, equation (35)) that $\nabla\cdot\overline{\boldsymbol{u}}^{\mathrm{L}} = 0$ correct to $O(a^2)$; and it is shown why the right-hand side of (1) is significant at leading order in the internal-gravity-wave problem referred to in §1.

Note, first, that Longuet-Higgins' proof assumes *inter alia*

(i) that the wave-amplitude field is steady; and

(ii) that to $O(a)$ the motion is one of small oscillations about rest, so $|\overline{\boldsymbol{u}}^{\mathrm{L}}| = O(a^2)$.

The right-hand side of (1) is then, indeed, negligible to $O(a^2)$. Second, if we follow Longuet-Higgins' basic assumptions and definitions, including the assumption (ii) just stated, we have $\partial\boldsymbol{\xi}/\partial t = \boldsymbol{u}$ and $\nabla\cdot\boldsymbol{\xi} = 0$ correct to $O(a)$. Furthermore $\overline{\boldsymbol{u}}^{\mathrm{L}} - \overline{\boldsymbol{u}} = \overline{(\boldsymbol{\xi}\cdot\nabla\boldsymbol{u})}$ correct to $O(a^2)$, as in the original problem considered by Stokes (1847),† so that $\nabla\cdot\overline{\boldsymbol{u}}^{\mathrm{L}} = \nabla\cdot\overline{(\boldsymbol{\xi}\cdot\nabla\boldsymbol{u})}$ correct to $O(a^2)$ since $\nabla\cdot\boldsymbol{u} = \nabla\cdot\overline{\boldsymbol{u}} = 0$. Now all these relations still hold if we relax the steadiness assumption (i) and regard $\overline{(\;)}$ as a running time average. This can be justified in the usual way, using the concept of two-timing or slow modulation that underlies the standard notions of wave packet, group velocity, and so on. It then follows that, correct to $O(a^2)$,

$$\nabla\cdot\overline{\boldsymbol{u}}^{\mathrm{L}} = \nabla\cdot\overline{(\boldsymbol{\xi}\cdot\nabla\boldsymbol{u})} = \overline{(\xi_j u_{i,j})_{,i}} = \overline{\xi_{j,i}u_{i,j}}$$

$$= \tfrac{1}{2}\overline{(\xi_{j,i}u_{i,j} + u_{j,i}\xi_{i,j})} = \frac{1}{2}\frac{\partial}{\partial t}\overline{(\xi_{j,i}\xi_{i,j})} = \frac{1}{2}\frac{\partial}{\partial t}\overline{(\xi_i\xi_j)}_{,ij}, \qquad \text{(A 1)}$$

where suffixes preceded by commas denote the corresponding spatial derivatives, $(\;)_{,i} = \partial(\;)/\partial x_i$. This directly verifies (1) under the assumption of slow modulations and the foregoing assumption (ii).

The two assumptions just stated hold true in the internal gravity waveguide problem referred to in §1. In that problem, a calculation of $\overline{\boldsymbol{u}}^{\mathrm{L}}(\boldsymbol{x}, t)$ from the Lagrangian-mean equations of motion would be in error at leading order if the right-hand side of (A 1) were to be inadvertently ignored. The problem is two-dimensional, and, correct to leading order for slow modulations, (A 1) reduces to

$$\frac{\partial\overline{u}^{\mathrm{L}}}{\partial x} + \frac{\partial\overline{w}^{\mathrm{L}}}{\partial z} = \frac{1}{2}\frac{\partial^3}{\partial t\,\partial z^2}\overline{(\zeta^2)}. \qquad \text{(A 2)}$$

The right-hand side of (A 2) is evidently non-zero with order of magnitude μa^2, where μ is the slow-modulation parameter, assumed small, that characterizes the temporal and horizontal spatial variability of averaged quantities. Because the wave packet has a lengthscale of order μ^{-1}, ignoring the right-hand side can be expected to produce an error of order μa^2 in $\partial\overline{u}^{\mathrm{L}}/\partial x$ and hence a^2 in $\overline{u}^{\mathrm{L}}$. That is, $\overline{u}^{\mathrm{L}}$ would be wrong to *leading* order. Detailed calculations like those presented in McIntyre (1973) confirm that expectation.

† This, also, depends on assumption (ii); if $\overline{\boldsymbol{u}}$ and $\overline{\boldsymbol{u}}^{\mathrm{L}}$ were $O(1)$ there would be a further $O(a^2)$ contribution $\frac{1}{2}\overline{\boldsymbol{\xi}\cdot(\boldsymbol{\xi}\cdot\nabla)\,\nabla\overline{\boldsymbol{u}}}$ to the Stokes drift defined as $\overline{\boldsymbol{u}}^{\mathrm{L}} - \overline{\boldsymbol{u}}$.

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