A General Theorem on Angular-Momentum Changes due to Potential Vorticity Mixing and on Potential-Energy Changes due to Buoyancy Mixing

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ABSTRACT

An initial zonally symmetric quasigeostrophic potential vorticity (PV) distribution $q_i(y)$ is subjected to complete or partial mixing within some finite zone $|y| < L$, where y is latitude. The change in M, the total absolute angular momentum, between the initial and any later time is considered. For standard quasigeostrophic shallow-water beta-channel dynamics it is proved that, for any $q_i(y)$ such that $dq_i/dy > 0$ throughout $|y| < L$, the change in M is always negative. This theorem holds even when "mixing" is understood in the most general possible sense. Arbitrary stirring or advective rearrangement is included, combined to an arbitrary extent with spatially inhomogeneous diffusion. The theorem holds whether or not the PV distribution is zonally symmetric at the later time. The same theorem governs Boussinesq potential-energy changes due to buoyancy mixing in the vertical. For the standard quasigeostrophic beta-channel dynamics to be valid the Rossby deformation length $L_D \gg \epsilon L$ where ϵ is the Rossby number; when $L_D = \infty$ the theorem applies not only to the beta channel but also to a single barotropic layer on the full sphere, as considered in the recent work of Dunkerton and Scott on "PV staircases." It follows that the M-conserving PV reconfigurations studied by those authors must involve processes describable as PV unmixing, or antidiffusion, in the sense of time-reversed diffusion. Ordinary jet self-sharpening and jet-core acceleration do not, by contrast, require unmixing, as is shown here by detailed analysis. Mixing in the jet flanks suffices. The theorem extends to multiple layers and continuous stratification. A least upper bound and greatest lower bound for the change in M is obtained for cases in which q_i is neither monotonic nor zonally symmetric. A corollary is a new nonlinear stability theorem for shear flows.

1. Introduction

Ideas about the turbulent mixing of vorticity and potential vorticity (PV), going back to the pioneering work of Taylor (1915, 1932), Dickinson (1969), Green (1970), and Welander (1973), are an important key to understanding such phenomena as Rossby-wave "surf zones," jet self-sharpening, and eddy-transport barriers. For a review see Dritschel and McIntyre (2008, hereafter DM08); also, for example, Killworth and McIntyre (1985), Hughes (1996), Held (2001), McIntyre (2008), Esler (2008a,b), and Bühler (2009). A key point is that PV mixing generically requires angular-momentum changes. In the real world those changes are usually mediated by, or catalyzed by, the radiation stresses or Eliassen–Palm fluxes due to Rossby waves and other wave types, including the form stresses exerted across undulating stratification surfaces. Usually, therefore, there is no such thing as turbulence without waves.

PV mixing by baroclinic and barotropic shear instabilities depends on radiation stresses internal to the system, mediating angular-momentum changes that add to zero. Cases like that of Jupiter's stratified weather layer probably depend on form stresses exerted from below, as is known to be true of the terrestrial stratosphere.

Consider for instance the quasigeostrophic thought experiment shown in Fig. 1a. This is an idealization of Rossby-wave surf-zone formation. An initially linear PV profile (thin line) is mixed such that the PV becomes uniform within a finite latitudinal zone $|y| < L$ (thick zigzag line). The mixing is assumed to be conservative in the sense that

$$\int\int dx\, dy\, \Delta q = 0, \tag{1.1}$$

where q is the quasigeostrophic PV and Δq its change due to mixing; $dxdy$ is the horizontal area element. It is

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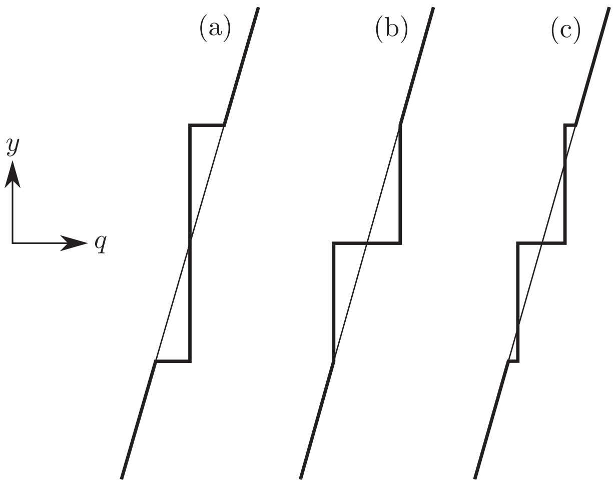

FIG. 1. Examples of initial and final zonally symmetric PV profiles (thin and thick lines, respectively). For each initial profile the PV increases linearly with latitude y. The examples could represent PV distributions in a quasigeostrophic shallow-water system, in a nondivergent barotropic system, or in a single layer within a multilayered or continuously stratified system. The angular-momentum changes ΔM are, respectively, negative, positive, and zero in cases (a)–(c), all of which satisfy (1.1). Cases (b) and (c) require unmixing, or antidiffusion. Dunkerton and Scott (2008) restrict attention to cases like (c).

well known that, according to standard quasigeostrophic theory, the resulting change ΔM in the total absolute angular momentum M is negative or retrograde, in this special case with the initial profile linear in y.[1] Such angular-momentum deficits are key to understanding why, for instance, breaking stratospheric Rossby waves gyroscopically pump a Brewer–Dobson circulation that is always poleward and never equatorward. The troposphere exerts a persistently westward form stress on the stratosphere. The physical reality of such surf-zone formation events and their tendency to mix PV has been verified in a vast number of observational and modeling studies, including studies of the stratospheric ozone layer (e.g., Lahoz et al. 2006, and references therein).

In an interesting recent paper in this journal, Dunkerton and Scott (2008, hereafter DS08), consider a class of PV reconfigurations in a single layer on the sphere, with zonally symmetric initial and final states, satisfying (1.1) and constructed so as to make $\Delta M = 0$. In DS08 the dynamics is nondivergent barotropic. That is, the Rossby deformation length $L_D = \infty$, and q is the absolute vorticity. As illustrated in DS08, the constraint (1.1) does not by itself dictate the sign of ΔM. However, in view of the ubiquity of radiation stresses in real atmospheres and oceans, one is led to question whether the assumption $\Delta M = 0$ is a natural one for realistic models.

Figure 1b shows a simple case where ΔM is positive and Fig. 1c a case where ΔM is zero as in DM08. Both these cases must involve unmixing, or antidiffusion. To go from the initial to the final state in Fig. 1b or Fig. 1c, one must transport q nonadvectively against its local gradient, at least in some locations (x, y). Such locally countergradient transport seems unnatural, at least as a persistent phenomenon in a model free of gravity wave stresses.

To exclude such countergradient transport we will restrict the PV reconfigurations, throughout this paper, not only to respect (1.1) but also to be describable as "generalized partial mixing," or "generalized mixing" for brevity. This will be made precise in section 2, using the standard "mixing kernel" or "redistribution function" formalism, but in essence it means that no unmixing is allowed. With that restriction, and a nonvanishing change Δq in the PV profile, we will prove a theorem stating that ΔM will always be negative, as it is in the special case of Fig. 1a, provided only that the initial PV profile is zonally symmetric and monotonically increasing in y. In all other respects the initial profile is arbitrary.

This theorem—which we designate as "basic" since it underpins the rest of our analysis—has been proved in several different ways. In section 5 we give what we think is the most readable of these proofs, after relating ΔM to Δq in sections 3 and 4. Section 6 points out that the basic theorem has an alternative interpretation in terms of potential energy and available potential energy.

Central to the proof in section 5 is an intrinsically nonnegative "bulk displacement function" constructed from the redistribution function. Its physical meaning is briefly discussed in section 7. Appendix A presents one of the alternative proofs, based on a second, quite different nonnegative function. That function is related to the so-called momentum–Casimir invariants of Hamiltonian theory and therefore mathematically related, also, to energy–Casimir invariants (e.g., Shepherd 1993) again connecting with the theory of available potential energy. This second nonnegative function is constructed from the initial PV profile rather than from the redistribution function.

The upshot is that from section 5 and appendix A we have two entirely different proofs not only of the sign definiteness of ΔM, but also of the sign definiteness of the potential-energy change due to generalized vertical mixing of an initially stable stratification. This generalizes classical results both on vortex dynamics (Arnol'd

[1] For an explicit demonstration, see, e.g., DM08 Eqs. (7.1)–(7.2) and below Eq. (A.4), noting that the integration by parts at the penultimate step is valid both for bounded and unbounded beta channels provided that the change $\Delta\overline{u}$ in the zonal-mean zonal flow vanishes at the side boundaries (Phillips 1954). For the unbounded channel, ΔM is entirely due to the ageostrophic mass shift associated with the northward residual circulation, since $\Delta\overline{u}$ integrates to zero. For the bounded channel there are contributions both from the mass shift and from $\Delta\overline{u}$.

1965) and on available potential energy (Holliday and McIntyre 1981), beyond the Hamiltonian framework. The potential-energy interpretation applies to a Boussinesq model with a linear equation of state. The proofs in sections 5 and appendix A provide us with two completely different types of sign-definite integral formulas, typified by (5.6) and (A.3) below, for ΔM and for the analogous sign-definite change in potential energy.

As summarized in section 8, the basic theorem covers three classes of model system: first, a shallow-water beta channel; second, a stratified quasigeostrophic beta channel; and third, the system considered in DS08—a sphere with $L_D = \infty$. Section 8 also points out that the basic theorem provides, as a corollary, a substantial generalization of the Charney–Stern shear-flow stability theorem, related also to the classical work of Arnol'd (1965).

Section 9 presents a generalization of the basic theorem to cases in which the initial PV profile is neither monotonic nor zonally symmetric.

Sections 10 and 11 discuss how the basic theorem applies to jet self-sharpening by PV mixing in the jet flanks. In section 10 we show via a specific example how a process for which ΔM must always be negative can nevertheless result in jet-core acceleration. Section 11 goes on to prove a much more general result. For the shallow-water model, PV mixing anywhere on one or both the flanks of a jet must always accelerate the jet core, provided that the jet is zonally symmetric both before and after mixing.

Section 12 briefly discusses the possibility of extending these results beyond quasigeostrophic to more accurate models. So far, we have failed to find such extensions. Obstacles to progress include the nonlinearity of accurate PV inversion operators. In the concluding remarks, section 13, we touch on the implications for models of geophysical turbulence. In particular, our results underline the need to pay closer attention to the angular-momentum budget in such models.

2. Definition of generalized mixing

As well as ordinary diffusion-assisted mixing we want to include the limiting case of purely advective rearrangement, or pure stirring. All such cases, from pure stirring to partial mixing to perfect mixing, can be described as linear operations on the PV field. They are conveniently represented in terms of a Green's function or integral kernel in the standard way (e.g., Pasquill and Smith 1983; Fiedler 1984; Stull 1984; Plumb and McConalogue 1988; Shnirelman 1993; Thuburn and McIntyre 1997; Esler 2008a). Such Green's functions have properties akin to probability density functions, and are called bistochastic or doubly stochastic. The corresponding linear operators are sometimes called polymorphisms.

The Green's function formalism is essentially the same for all the model systems under consideration, including those describing potential-energy changes. So it will suffice to restrict attention at first to the shallow-water case. For a general two-dimensional domain $\mathcal{D}$, let $q_i(x, y)$ be the initial PV distribution and $q_\ell(x, y)$ the PV distribution at some later time. Because of linearity and horizontal nondivergence we may write

$$q_\ell(x, y) = \iint_{\mathcal{D}} dx'\, dy'\, q_i(x', y') r(x', y'; x, y) \tag{2.1}$$

where the kernel r satisfies the following three conditions:

$$\iint_{\mathcal{D}} dx\, dy\, r(x', y'; x, y) = 1 \quad \text{for all} \quad (x', y') \in \mathcal{D}, \tag{2.2}$$

$$\iint_{\mathcal{D}} dx'\, dy'\, r(x', y'; x, y) = 1 \quad \text{for all} \quad (x, y) \in \mathcal{D}, \tag{2.3}$$

$$\text{and} \quad r(x', y'; x, y) \geq 0 \quad \text{for all} \quad (x, y), (x', y') \in \mathcal{D}, \tag{2.4}$$

but is otherwise arbitrary. Here we call $r(x', y'; x, y)$ the "redistribution function" defining the generalized mixing that takes place between the initial time and the later time. The condition (2.2) ensures that $r(x', y'; x, y)$ represents a conservative redistribution of PV substance in the sense that (1.1) is satisfied. To show this, integrate (2.1) with respect to x and y and then use (2.2) to deduce (1.1) with $\Delta q = q_\ell - q_i$. The conditions (2.3) and (2.4) ensure that, for given (x, y), $q_\ell(x, y)$ is a weighted average (with positive or zero weights) of the initial PV values $q_i(x', y')$. This in turn ensures that generalized mixing cannot increase the range of PV values, and in particular that an initially uniform PV profile remains uniform.

We may think of $r(x', y'; x, y)\, dxdydx'dy'$ as the proportion of fluid transferred from area $dx'dy'$ at location (x', y') to area $dxdy$ at location (x, y). Here "fluid" has to be understood in a particular way. The notional fluid, or material, has to be the sole transporter of q substance, whether by advection or by diffusion or otherwise. That is, we imagine that different amounts of q substance are attached permanently to each fluid particle so that, in particular, the diffusivity of q is the same as the self-diffusivity of the notional fluid. The notional fluid is incompressible, as required by (1.1), (2.2), and the concept of self-diffusivity.

The mathematical properties of the Green's function operators are further discussed in Shnirelman (1993).

For instance, they form a partially ordered semigroup. The partial ordering corresponds to successive mixing events.

For the PV-mixing problem we are mainly interested in a zonally symmetric domain $|y'| < L$; and sections 2–8 will consider only zonally symmetric initial PV profiles, $q_i(y')$. The PV distribution after generalized mixing may or may not be zonally symmetric. However, the angular-momentum change ΔM depends only on $q_i(y')$ and on the zonal or x average of $q_\ell(x, y)$, denoted $\overline{q}_\ell(y)$. It is convenient to define

$$R(y', y) := \int dx'\, \overline{r(x', y'; x, y)}, \tag{2.5}$$

the overbar again denoting the average with respect to x (not x'). The zonal average of (2.1) is then

$$\overline{q}_\ell(y) = \int_{-L}^{L} dy'\, q_i(y') R(y', y), \quad \text{where} \tag{2.6}$$

$$\int_{-L}^{L} dy\, R(y', y) = 1 \quad \text{for all} \quad y' \in [-L, L], \tag{2.7}$$

$$\int_{-L}^{L} dy'\, R(y', y) = 1 \quad \text{for all} \quad y \in [-L, L], \quad \text{and} \tag{2.8}$$

$$R(y', y) \ge 0 \quad \text{for all} \quad y, y' \in [-L, L], \tag{2.9}$$

(2.7)–(2.9) being the counterparts of (2.2)–(2.4).

A redistribution function R representing pure diffusion is symmetric in the sense that $R(y', y) = R(y, y')$. This follows from the self-adjointness of the operator representing the divergence of a downgradient diffusive flux. It is sometimes assumed that all redistribution functions are symmetric, but that would be too restrictive for our purposes.

Consider the examples of purely advective rearrangement in Fig. 2. The first two examples, with redistribution function $R_1(y', y)$ and $R_2(y', y)$, are symmetric. They correspond to patterns in the (y', y) plane that are mirror symmetric about the main diagonal, representing simple pairwise diffusionless exchanges of fluid elements. The third example depicts the effect of R_1 followed by R_2, giving

$$\overline{q}_\ell(y) = \int_{-L}^{L} dy' \int_{-L}^{L} dy''\, q_i(y'') R_1(y'', y') R_2(y', y). \tag{2.10}$$

That is, the effect of R_1 followed by R_2 is described by the composite redistribution function

$$R_2 \circ R_1(y'', y) := \int_{-L}^{L} dy'\, R_1(y'', y') R_2(y', y), \tag{2.11}$$

which is asymmetric. It represents a cyclic permutation of three fluid elements and is the simplest kind of asymmetric redistribution function. To be completely general we need to include such cases and their elaborations.

In section 9 and appendix A we use the fact that purely advective rearrangements are reversible, hence described by invertible mappings.

3. *M* in terms of *q* for shallow water

For shallow-water beta channel dynamics we may define M as the total absolute zonal momentum per unit zonal (x) distance. Let the shallow-water layer have depth $H - b(x, y, t) + h(x, y, t)$, where H is constant, h is the free surface elevation, b is the bottom topography, and $h \ll H$, $b \ll H$. We assume $\overline{b} = \overline{b}(y)$. The fluctuating part $\tilde{b}(x, y, t) := b - \overline{b}$ can provide a quasi-topographic form stress to change M and catalyze PV mixing, as may happen in Jupiter's stratified weather layer. We choose the Coriolis parameter to be a constant, f_0, thus regarding the beta effect as due to the northward or y gradient of the zonally averaged bottom profile $\overline{b}(y)$, corresponding to the latitudinal gradient of Taylor–Proudman layer depth in the middle latitudes of a spherical planet. Let ρ_0 be the constant mass density and $u(x, y, t)$ the zonal velocity with $\overline{u}(y, t)$ its zonal average. Then to quasi-geostrophic accuracy

$$M = \rho_0 \int_{-L}^{L} dy\, \overline{(H + h - b)(u - f_0 y)} \tag{3.1}$$

$$= \rho_0 H \int_{-L}^{L} dy \left(\overline{u} - f_0 y \frac{\overline{h} - \overline{b}}{H} - f_0 y \right) \tag{3.2}$$

$$= \rho_0 H \int_{-L}^{L} dy \left(\overline{u} - f_0 y \frac{\overline{h} - \overline{b}}{H} \right) + \text{const.} \tag{3.3}$$

Introducing the quasigeostrophic streamfunction $\psi = gh/f_0$ and the Rossby deformation length $L_D = \sqrt{gH}/f_0$ we have

$$M = \rho_0 H \int_{-L}^{L} dy \left(-\frac{\partial \overline{\psi}}{\partial y} - f_0 y \frac{\overline{h} - \overline{b}}{H} \right) + \text{const.} \tag{3.4}$$

$$= \rho_0 H \int_{-L}^{L} dy \left(\frac{\partial^2 \overline{\psi}}{\partial y^2} - L_D^{-2} \overline{\psi} + \beta y \right) y + \text{const.}, \tag{3.5}$$

where the first term has been integrated by parts. We have defined

$$\beta y := \frac{f_0 \overline{b}}{H} \tag{3.6}$$

and assumed that the Phillips boundary condition holds,

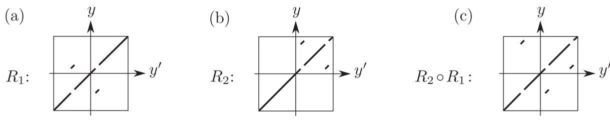

FIG. 2. Three redistribution functions of which the first two are symmetric and the third asymmetric. They represent purely advective rearrangements within the zone $|y| < L$, with no diffusive smearing. Values are zero except on the black sloping lines, which represent Dirac delta functions, e.g., $\delta(y - y')$ on the main diagonal $y = y'$. The first two redistribution functions R_1 and R_2 describe simple exchanges of small but finite (striplike) fluid elements. The composite rearrangement described by the third redistribution function $R_2 \circ R_1$, see (2.11), is a cyclic permutation among three fluid elements, a "three-cycle" in group-theoretic terminology. Notice that the off-diagonal delta functions line up with the gaps, or zeros, in the main diagonal. They line up both in the y direction and in the y' direction so that both (2.7) and (2.8) are satisfied.

namely

$$\frac{\partial \overline{u}}{\partial t} = -\frac{\partial^2 \overline{\psi}}{\partial y \partial t} = 0 \quad \text{on} \quad y = \pm L, \tag{3.7}$$

implying that the boundary term

$$-\rho_0 H \left[\frac{\partial \overline{\psi}}{\partial y} y\right]_{-L}^{+L} = \text{const.} \tag{3.8}$$

The Phillips boundary condition is the standard way of stopping mass and angular momentum from leaking across the side boundaries (Phillips 1954). Denoting the variable part of M in (3.5) by $\tilde{M}$ and defining q in the standard way, ignoring a contribution f_0, as

$$q := \nabla^2 \psi - L_D^{-2} \psi + \beta y, \tag{3.9}$$

we have

$$\tilde{M} = \rho_0 H \int_{-L}^{L} dy\, \overline{q}(y) y. \tag{3.10}$$

This expression has an alternative interpretation as the *Kelvin impulse* for the quasigeostrophic system, per unit zonal distance (e.g., Bühler 2009). Initially

$$\tilde{M} = \tilde{M}_i := \rho_0 H \int_{-L}^{L} dy\, q_i(y) y. \tag{3.11}$$

At the later time after generalized mixing, the averaged q becomes $\overline{q}_\ell = q_i + \Delta \overline{q}$, so that

$$\tilde{M} = \tilde{M}_\ell := \rho_0 H \int_{-L}^{L} dy\, \overline{q}_\ell(y) y, \tag{3.12}$$

$$= \rho_0 H \int_{-L}^{L} dy \int_{-L}^{L} dy'\, q_i(y') R(y', y) y, \tag{3.13}$$

$$= \tilde{M}_i + \Delta M, \tag{3.14}$$

say, with

$$\Delta M = \rho_0 H \int_{-L}^{L} dy\, \Delta \overline{q}(y) y, \tag{3.15}$$

$$= \rho_0 H \int_{-L}^{L} dy \int_{-L}^{L} dy'\, q_i(y') \Delta R(y', y) y, \tag{3.16}$$

where

$$\Delta R(y', y) := R(y', y) - \delta(y' - y), \tag{3.17}$$

the difference between the redistribution function $R(y', y)$ and the do-nothing redistribution function $\delta(y' - y)$. Here δ denotes the Dirac delta function.

4. *M* in terms of *q* for other systems

The relations in section 3 extend straightforwardly to the sphere and to a stratified quasigeostrophic system in a beta channel.

In the stratified system, with say a bottom boundary at pressure altitude $z = z_0$, the PV is redistributed separately on each z surface, and the buoyancy acceleration $f_0 \partial \psi / \partial z$ is redistributed on $z = z_0$. Therefore, each altitude z has its own R and ΔR functions, $R(y', y; z)$ and $\Delta R(y', y; z)$ say. To obtain a concise formulation we may define the PV to include a delta function at $z = z_0$ following Bretherton (1966),

$$Q(x, y; z) := \nabla^2 \psi + \frac{1}{\rho_0(z)} \frac{\partial}{\partial z} \left(\rho_0(z) \frac{f_0^2}{N(z)^2} \frac{\partial \psi}{\partial z} \right) + \frac{f_0^2}{N(z_0)^2} \frac{\partial \psi}{\partial z} \delta(z - z_0) + \beta y, \tag{4.1}$$

where $\rho_0(z)$ is the background density, $N(z)$ is the background buoyancy frequency, and ∇^2 still denotes the horizontal Laplacian. If there is a rigid top boundary, then a further delta function can be added. The initial and later $\tilde{M}$ values and the difference between them are now, respectively,

$$\tilde{M}_i = \int dz \rho_0(z) \int_{-L}^{L} dy\, Q_i(y; z) y, \tag{4.2}$$

$$\tilde{M}_\ell = \int dz \rho_0(z) \int_{-L}^{L} dy \int_{-L}^{L} dy'\, Q_i(y'; z) R(y', y; z) y, \quad \text{and} \tag{4.3}$$

$$\Delta \tilde{M} = \int dz\, \rho_0(z) \int_{-L}^{L} dy \int_{-L}^{L} dy'\, Q_i(y'; z) \Delta R(y', y; z) y. \tag{4.4}$$

The contributions to $\tilde{M}$ add up layerwise because PV inversion is a linear operation in quasigeostrophic dynamics.

These relations also extend to a single layer on a sphere, provided that $L_D = \infty$ and that absolute zonal momentum per unit zonal distance is replaced by absolute angular momentum per radian of longitude. Then the counterpart of (3.15) is

$$\Delta M = \rho_0 H a^4 \int_{-1}^{1} d\mu\, \Delta\overline{q}(\mu)\mu, \tag{4.5}$$

where a is the radius of the sphere, $\mu := \sin\phi$ where ϕ is the latitude, and q now denotes the absolute vorticity. The generalized-mixing conditions (2.6)–(2.9) and the formulas (3.11)–(3.17) apply to the sphere provided that y is replaced by μ, $\rho_0 H$ by $\rho_0 H a^4$, and $\pm L$ by ± 1.

5. The basic theorem

In this section we prove the basic theorem that ΔM is always negative for monotonically increasing $q_i(y')$ and any nontrivial rearrangement function R such that integrals like (3.16) make mathematical sense, with values independent of the order of integration. The same proof will apply to the potential-energy problem, with zonal averaging replaced by horizontal area integration for general container shapes, as explained in section 6.

Nontrivial means "do something" rather than "do nothing": ΔR in (3.17) must be nonvanishing in an appropriate sense. More precisely, nontrivial means that $R(y', y)$ and $\Delta R(y', y)$ have nonvanishing off-diagonal values somewhere, where those off-diagonal values have *nonzero measure* in the sense that they can make nonzero contributions to integrals like (3.16). This in turn means that the nonvanishing off-diagonal values must exist in some finite neighborhood, albeit possibly a neighborhood in the form of a line segment, as in the delta-function examples of Fig. 2.

Equation (3.16) can be rewritten

$$\Delta M = \rho_0 H \int_{-L}^{L} dy'\, q_i(y')\eta(y'), \tag{5.1}$$

where by definition

$$\eta(y') := \int_{-L}^{L} dy\, \Delta R(y', y) y. \tag{5.2}$$

By virtue of (3.17), $\eta(y')$ may be regarded as the average latitudinal displacement of fluid initially at y'. Denote the indefinite integral of $\eta(y')$ by $И(y')$ (Cyrillic-style capital Eta). Specifically,

$$И(y') := \int_{-L}^{y'} dy''\, \eta(y'') = \int_{-L}^{y'} dy'' \int_{-L}^{L} dy\, \Delta R(y'', y) y \tag{5.3}$$

$$= \int_{-L}^{y'} dy'' \int_{-L}^{L} dy\, \Delta R(y'', y)(y - y') \tag{5.4}$$

$$= -\int_{y'}^{L} dy'' \int_{-L}^{L} dy\, \Delta R(y'', y)(y - y'), \tag{5.5}$$

where the penultimate step uses (2.7) and (3.17), implying that $\int_{-L}^{L} dy\, \Delta R(y'', y) = 0$ for all y'', and the last step (2.8) and (3.17), implying that $\int_{-L}^{L} dy''\, \Delta R(y'', y) = 0$ for all y. The last step depends on interchangeability of the order of integration. From (5.4) and (5.5) we see that $И(-L) = 0 = И(+L)$. Therefore (5.1) may be integrated by parts to give

$$\Delta M = -\rho_0 H \int_{-L}^{L} dy'\, \frac{\partial q_i(y')}{\partial y'} И(y'). \tag{5.6}$$

So if, finally, for nontrivial R, we can prove that $И(y')$ is nonnegative for all values of y' and nonvanishing with nonzero measure for at least some values of y', then the theorem will follow. That is, (5.6) will then imply that

$$\Delta M < 0 \quad \text{if} \quad \frac{\partial q_i(y')}{\partial y'} > 0 \quad \text{for all} \quad y', \tag{5.7}$$

and vice versa. That is, the sign of ΔM must always be opposite to the sign of the initial monotonic PV gradient.

To prove that $И(y')$ is nonnegative we rewrite (5.5), after changing the order of integration, as

$$И(y') = \int_{-L}^{y'} dy \int_{y'}^{L} dy''\, \Delta R(y'', y)|y - y'| - \int_{y'}^{L} dy \int_{y'}^{L} dy''\, \Delta R(y'', y)|y - y'|. \tag{5.8}$$

Again because $\int_{-L}^{L} dy''\, \Delta R(y'', y) = 0$ for all y, we may replace $\int_{y'}^{L} dy''$ by $-\int_{-L}^{y'} dy''$. Applying this to the second term only, we obtain an expression in which ΔR can be replaced by the nonnegative function R,

$$И(y') = \int_{-L}^{y'} dy \int_{y'}^{L} dy''\, R(y'', y)|y - y'| + \int_{y'}^{L} dy \int_{-L}^{y'} dy''\, R(y'', y)|y - y'|, \tag{5.9}$$

because there are no contributions from the main diagonal $y = y''$. For given y', the two rectangular domains of integration for (5.9) intersect each other and the main diagonal at a single point only, $y = y'' = y'$. (The two domains are mirror images of each other in the main diagonal.) At the point $y = y'' = y'$ the factor $|y - y'|$ is zero, annihilating any delta functions. Therefore $И(y')$ is nonnegative.

Now as y' runs from $-L$ to L, the two domains sweep over the upper and lower triangles of the square $-L \le y \le L$, $-L \le y'' \le L$, together covering the entire square. By definition, a nontrivial R function must have nonzero measure somewhere off the main diagonal, in some finite neighborhood of a location with $|y| \ne L$, $|y'| \ne L$, and $y - y' \ne 0$. Whichever moving domain encounters that location must continue to intersect it as y' runs through some finite range of values, implying that $И(y') > 0$ over that finite range. So $И(y')$ is not only nonnegative, but also nonvanishing with nonzero measure, for any nontrivial R, and the theorem follows.

An alternative proof using an entirely different nonnegative function is given in appendix A.

6. Connection to available potential energy

The basic theorem can alternatively be read as governing the sign of the potential-energy change due to three-dimensional generalized mixing of a Boussinesq fluid within a fixed container in a uniform gravitational field.

Consider first a container with vertical walls. Then (5.6) carries over at once if we read q_i as the buoyancy acceleration, y' as the altitude, the R and ΔR functions as applying to horizontal area averages, starting from a three-dimensional version of the r function in (2.1)–(2.4), and ΔM as proportional to minus the potential-energy change. Second, consider a container of arbitrary shape $\mathcal{V}$ as being embedded within the vertical-walled container. We merely extend the definition of r and hence of R and ΔR such that no generalized mixing takes place outside $\mathcal{V}$. With this understanding (5.6) still applies, and (5.7) follows. That is, if the initial state is undisturbed and stably stratified, with the same stratification at all horizontal positions (including those in any separate ''abyssal basins''), then the potential-energy change is guaranteed to be positive for any nontrivial R whatever. This generalizes a standard result in the theory of available potential energy saying the same thing for a purely advective R (e.g., Holliday and McIntyre 1981; appendix A below).

We emphasize that the generalized result depends on having a linear equation of state, as is standard for Boussinesq models, since only then is the buoyancy acceleration a transportable, mixable quantity.[2]

[2] For more general equations of state, especially those containing thermobaric terms, there is no straightforward concept of potential energy. As first shown by W. R. Young, the Boussinesq limit then needs reconsideration, and the consequences are nontrivial. It turns out that potential energy has to be replaced by a ''dynamic enthalpy'' that contains both gravitational and vestigial thermodynamic contributions (Young 2010). Such generalized Boussinesq models are outside our scope here.

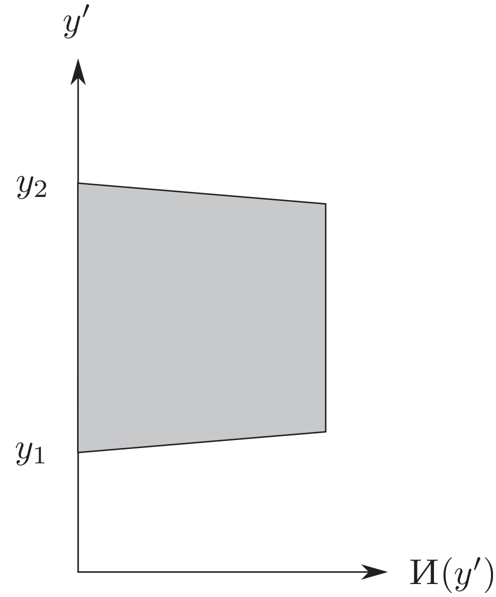

FIG. 3. Illustration of $И(y')$ for a redistribution function that simply exchanges material between latitudes y_1 and y_2, as for instance in Fig. 2b. The finite slopes near y_1 and y_2 are due to the finite widths of the fluid elements exchanged. The maximum value of $И$ is $y_2 - y_1$.

7. The physical meaning of $И(y')$

Reverting to the PV interpretation, with y northward rather than upward, we consider the function $И(y')/(L + y')$. The definition (5.3) shows that $И(y')/(L + y')$ is the average northward displacement of all the notional fluid initially south of y'. Equivalently, $И(y')/(L + y')$ is the northward displacement of that fluid's centroid. This makes the nonnegativeness of $И$ more intuitively apparent. The centroid is initially as far south as it can be, and can therefore only move northward. We may reasonably call $И(y')$ itself the ''area-weighted bulk displacement'' of all the fluid initially south of y', or ''bulk displacement function'' for brevity.

The fact that $И(L) = 0$ expresses what can also, now, be seen to be intuitively reasonable, namely, that there can be no bulk displacement of the entire zone $-L \le y' \le L$. The fluid has nowhere to go. Its centroid must remain fixed under any generalized mixing operation confined to the zone $-L \le y' \le L$. And the symmetry expressed by (5.9) says that we may equally well think of $И(y')$ as the *southward* area-weighted bulk displacement of all the fluid initially *north* of $y = y'$.

Figure 3 shows a simple example, the bulk displacement function $И(y')$ corresponding to the R function shown in Fig. 2b. Nothing happens to the fluid south of y_1 and north of y_2. However, there is, for instance, a northward bulk displacement of the fluid originally in

$(-L, y')$ whenever y' lies between y_1 and y_2. The transitions across y_1 and y_2 have small but finite widths, corresponding to the small but finite line segments in the off-diagonal regions of Fig. 2b.

The foregoing applies of course to the potential-energy interpretation, with northward and southward replaced by upward and downward.

8. Further implications, including generalized shear-instability theorems

The result (5.6) carries over to DS08's case of a sphere with $L_D = \infty$, with y replaced by μ, the sine of the latitude, and ΔM replaced by its spherical counterpart, the absolute angular-momentum increment (4.5), as noted at the end of section 4. And (5.6) also carries over to the stratified systems of section 4, with the factor $\rho_0 H$ replaced by a vertical integration and q by Q as in (4.1)–(4.4). Therefore, the basic theorem (5.7) holds in any case for which there are monotonic profiles of Q on each of the levels subject to mixing, provided that all the gradients $\partial Q/\partial y$ have the same sign including the gradients of the Bretherton delta function or functions.

It is worth noting the implications of such cases for the theory of quasigeostrophic shear instability, in particular the theorems of Charney and Stern (1962) and Arnol'd (1965). These theorems in their original forms apply only to nondiffusive Hamiltonian dynamics, and therefore only to purely advective rearrangements. The basic theorem (5.7) generalizes the Charney–Stern theorem and a case of Arnol'd's first stability theorem—which we call ''Arnol'd's zeroth stability theorem,'' or ''the Arnol'd theorem,'' for brevity—to cover finite-amplitude disturbances with arbitrary amounts of PV mixing. The Arnol'd theorem in question is the nonlinear counterpart of the Rayleigh–Kuo theorem, rather than the Fjørtoft theorem of which Rayleigh–Kuo is a special case.

In instability problems there are no external sources or sinks of absolute angular momentum. Growing instabilities exchange angular momentum purely internally, through radiation or diffraction stresses. This is possible, the basic theorem tells us, only if there are regions in which the q or Q gradients have different signs. Conversely, whenever the q or Q gradients are nonzero and all of one sign, instability is impossible. These are exactly the circumstances in which the Charney–Stern theorem and the Arnol'd theorem were originally proved for purely advective rearrangements and can now be proved, using (5.7), for the far more general redistributions defined in section 2, which include PV mixing.

The proof runs as follows. We start with $q = q_i(y)$, or $Q = Q_i(y)$ on each level. An initial finite-amplitude disturbance is set up advectively, by undulating the PV contours. To do so requires artificial forcing. This is because of the hypothesis that the q or Q gradients are nonzero and all of one sign. By (5.7), M must change by some nonvanishing amount ΔM during the setup.

We then let the system run freely. The free dynamical evolution may include wave breaking and PV mixing—going beyond Hamiltonian evolution. PV invertibility implies that the free evolution can be fully described by specifying a succession of PV distributions. Equivalently, therefore, the free evolution can be described by a succession of R functions operating on $q = q_i(y)$ or $Q = Q_i(y)$. Each such function is the composite of two R functions, the purely advective R function describing the initial setup and one of the general R functions describing the subsequent free evolution.

The free evolution keeps ΔM constant. Since (5.6) or its Q counterpart, vertically integrated as necessary, is sign definite by hypothesis, either it or its negative qualifies as a Lyapunov function (from R functions to nonnegative real numbers), whose constancy under free evolution implies neutral nonlinear stability. This is the generalized Arnol'd's zeroth theorem.

We may remark that the sign-definite function (A.3) below also qualifies as a Lyapunov function, vertically integrated as necessary, providing an alternative proof.

9. Nonmonotonic, zonally asymmetric q_i

The basic theorem (5.7) applies to zonally symmetric and monotonic $q_i(y')$ only. This is the most important case, but it may be of interest to note what can be proved for more general initial conditions $q_i(x', y')$.

Consider a pair of PV distributions $q_1(x', y')$, $q_2(x', y')$ that can be derived from each other by purely advective, and therefore reversible, rearrangement. That is,

$$q_2(x, y) = \int\int_{\mathcal{D}} dx' dy' q_1(x', y') s(x', y', x, y) \quad \text{and} \tag{9.1}$$

$$q_1(x, y) = \int\int_{\mathcal{D}} dx'\, dy'\, q_2(x', y') s(x, y, x', y'), \tag{9.2}$$

where the redistribution function s describes an invertible mapping.

For given s, consider the set of all possible redistribution functions r together with the set of all possible composites $r \circ s$. Because of reversibility, the set of all r must be the same as the set of all $r \circ s$. Therefore the set of all possible $\tilde{M}_\ell$ values that can result from applying the r's to an initial PV distribution q_1 must be the same as the set of all possible $\tilde{M}_\ell$ values from applying the r's to an initial q_2.

For a general initial $q_1(x', y')$ we can always find an advective rearrangement s such that q_2 is a monotonically increasing function of y alone (appendix B). Denote that function by $q_2[y; q_1(\cdot)]$. The corresponding $\tilde{M}$ value is

$$\tilde{M}_2[q_1(\cdot)] = \int_{-L}^{L} dy'\, q_2[y'; q_1(\cdot)]y'. \tag{9.3}$$

The basic theorem of section 5 restricts the possible $\tilde{M}_\ell$ values that can be attained starting from $q_2[y; q_1(\cdot)]$. Specifically,

$$\tilde{M}_\ell \le \tilde{M}_2[q_1(\cdot)]. \tag{9.4}$$

The same argument applies to the monotonically decreasing case. Because the y origin is in the center of the y domain, the resulting q_2 function is simply $q_2[-y; q_1(\cdot)]$ and the corresponding $\tilde{M}$ value is $-\tilde{M}_2[q_1(\cdot)]$. In summary, identifying $q_1(x', y')$ with our general initial condition $q_i(x', y')$, we now have

$$-\tilde{M}_2[q_i(\cdot)] \le \tilde{M}_\ell \le \tilde{M}_2[q_i(\cdot)]. \tag{9.5}$$

That is, the two possible extreme values of $\tilde{M}_\ell$ correspond to the two extreme, monotonically decreasing or increasing, zonally symmetric profiles into which $q_i(x', y')$ can be advectively rearranged.

10. The simplest jet-resharpening problem

Consider the following shallow-water thought experiment in an unbounded domain, $L = \infty$. We begin with a perfectly sharp jet, with concentrated PV gradients at its core (solid curves in Fig. 4). First, the concentrated PV gradients are smeared out, decelerating the jet and decreasing the absolute angular momentum M (dotted curves in Fig. 4). Second, the PV is mixed on both sides of the jet, resharpening and accelerating it (dashed curves in Fig. 4). Perhaps counterintuitively, the basic theorem (5.7) implies that M must decrease further, at this second stage, even though the jet core accelerates. Let us look at what happens in more detail.

Consider the quasigeostrophic shallow-water system with the initial PV profile in the form of a step of size $2q_s$,

$$q_i(y') = \begin{cases} q_s & (y' > 0) \\ -q_s & (y' < 0). \end{cases} \tag{10.1}$$

Inversion gives the familiar velocity profile

$$u_i(y') = q_s L_D \exp\left(\frac{-|y'|}{L_D}\right), \tag{10.2}$$

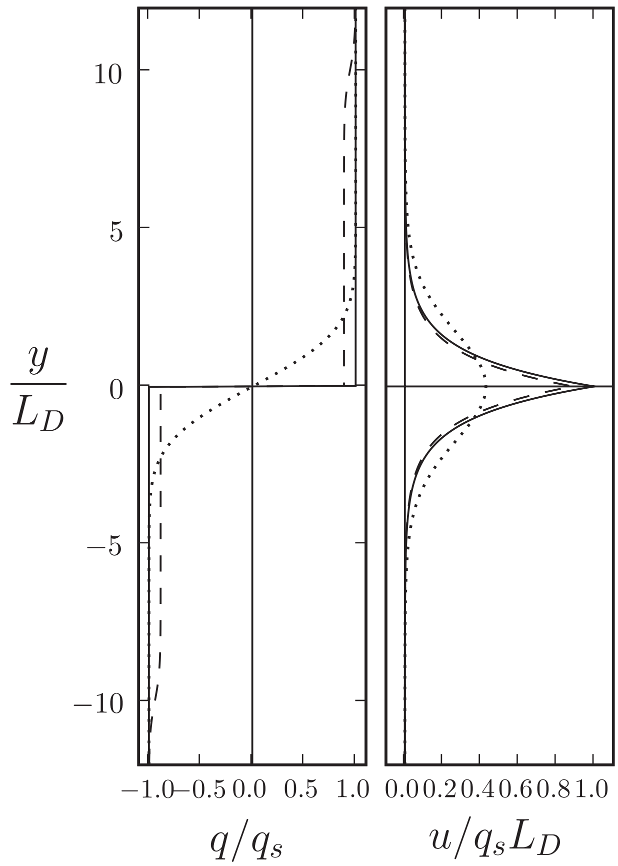

FIG. 4. The jet-resharpening thought experiment. An initial PV profile in the form of a step function (solid curves) is smeared diffusively (dotted curves). This smeared profile is then resharpened by mixing PV on the flanks of the jet (dashed curves). Here, as throughout this paper, "mixing" entails conservation of PV substance (1.1).

shown by the solid curve in Fig. 4b. After the first stage, in which the concentrated gradients are smeared out, the PV profile is taken in error-function form

$$q_1(y) = \frac{2q_s}{L\sqrt{\pi}} \int_0^y d\tilde{y}\, \exp\left(-\frac{\tilde{y}^2}{L^2}\right), \tag{10.3}$$

shown by the dotted curve in Fig. 4a, for which the length scale L has been taken as $2L_D$. Inversion gives the corresponding smeared velocity profile, shown by the dotted curve in Fig. 4b, as

$$u_1(y) = \frac{L_D q_s}{L\sqrt{\pi}} \int_{-\infty}^{\infty} d\tilde{y}\, \exp\left(-\frac{|y-\tilde{y}|}{L_D} - \frac{\tilde{y}^2}{L^2}\right) \tag{10.4}$$

[cf. (11.2)ff.]. The change in M due to the PV redistribution in this first stage is

$$\Delta M_1 = \int_{-\infty}^{\infty} dy\, (q_1 - q_i)y = -\frac{q_s L^2}{2}, \tag{10.5}$$

as can be verified from an integration by parts.

After the second stage, the PV has been perfectly mixed on either side of the jet core (dashed curves) out to fringes at around $|y| = \lambda L_D$, say, where $\lambda \gg 1$. In the figure, we have taken $\lambda = 10$. Within the two perfectly mixed regions, the resharpened PV distribution is

$$q_2(y') = \begin{cases} q_s - \delta q_s & (y' > 0) \\ -(q_s - \delta q_s) & (y' < 0) \end{cases}, \quad \text{where} \tag{10.6}$$

$$\delta q_s = (\lambda L_D)^{-1}\left(\frac{q_s L}{\sqrt{\pi}}\right) \ll q_s. \tag{10.7}$$

This assumes fringes antisymmetric about $y = \pm\lambda L_D$, as well as total PV conservation, Eq. (1.1), and neglect of the Gaussian tails in (10.3) for $y \gg L_D$. The corresponding resharpened velocity profile is

$$u_2(y') = (q_s - \delta q_s) L_D \exp(-|y'|/L_D), \tag{10.8}$$

provided that the peripheral fringes have length scales $\gg L_D$. (Narrower peripheral fringes, not $\gg L_D$, would invert to give two extra jets, albeit weak ones.)

The change in M due to the PV redistribution in the second stage is

$$\begin{aligned} \Delta M_2 &= -\Delta M_1 + \int_{-\lambda L_D}^{\lambda L_D} dy (q_2 - q_i) y \\ &= -q_s L^2 \left\{\lambda\left(\frac{L_D}{L\sqrt{\pi}}\right) - \frac{1}{2}\right\}. \end{aligned} \tag{10.9}$$

Because $\lambda \gg 1$, $|\Delta M_2| \gg |\Delta M_1|$. In this example, M not only decreases at each stage, as the basic theorem says it must, but the decrease is far greater at the second stage, even though the jet core still accelerates. The total change ΔM over both stages is

$$\Delta M = \Delta M_1 + \Delta M_2 = -q_s L^2 \lambda\left(\frac{L_D}{L\sqrt{\pi}}\right). \tag{10.10}$$

This can also be written

$$\Delta M = -\frac{q_s L^2}{\pi}\left(\frac{\delta q_s}{q_s}\right)^{-1}. \tag{10.11}$$

As $\delta q_s/q_s$ decreases, the PV profile q_2 returns closer and closer to the initial PV profile q_i while ΔM becomes increasingly large and negative. There is an increasingly large cost associated with mixing far from $y = 0$. In the case of Fig. 4, $\delta q_s/q_s \approx 0.11$. Furthermore, $\Delta M_1 = -2q_s L_D^2$ and $\Delta M_2 \approx -9q_s L_D^2$. If $\delta q_s/q_s$ were decreased to 0.01, then ΔM_2 would become $\approx -125 q_s L_D^2$.

11. General jet sharpening

Consider a more general shallow-water thought experiment, now starting from a general monotonic PV profile $q_i(y')$. For definiteness, we take the monotonically increasing case $\partial q_i(y')/\partial y' > 0$.

We suppose that generalized mixing takes place except that there is no mixing across a particular material contour initially at latitude $y = y_0$. That is, the contour behaves as an eddy-transport barrier. The contour may undulate during the mixing, but we assume that it straightens out afterward and returns to latitude y_0, consistent with quasigeostrophic, area-preserving advection. The net effect of the mixing can then be described by a nontrivial zonally averaged redistribution function $R(y', y) = R(y', y; y_0)$, recall (2.5), such that

$$R(y', y; y_0) = 0 \;\text{ if }\; y < y_0 < y' \;\text{ or }\; y' < y_0 < y. \tag{11.1}$$

It will be proved that, in this thought experiment, for finite L_D, the net change $\Delta\overline{u}(y_0)$ in the zonal-mean zonal flow at $y = y_0$ is always positive.

In particular, we may choose the material contour $y = y_0$ to be in the core of a jet. So *mixing on one or both flanks of a jet must always accelerate the straightened-out jet core, regardless of the details of the mixing provided only that the jet core has persisted, throughout, as an eddy-transport barrier.* Mixing could be confined, for instance, to locations arbitrarily far from the jet core, though of course the resulting $\Delta\overline{u}(y_0)$ would then be small.

Differentiating the expression for q in (3.9) with respect to y, and taking the zonal average, we obtain the inversion problem for the change $\Delta\overline{u}(y)$ in $\overline{u}(y)$ due to an arbitrary change $\Delta\overline{q}(y)$ in $\overline{q}(y)$,

$$\left(\frac{\partial^2}{\partial y^2} - L_D^{-2}\right)\Delta\overline{u}(y) = -\frac{\partial \Delta\overline{q}(y)}{\partial y}. \tag{11.2}$$

This can be solved with the Green's function $G(y, y_0)$ defined by

$$\left(\frac{\partial^2}{\partial y^2} - L_D^{-2}\right) G(y, y_0) = -\delta(y - y_0) \tag{11.3}$$

with $G(y, y_0)$ vanishing on the boundaries $y \pm L$ to satisfy the Phillips boundary condition (3.7). The proof will apply both to finite and to infinite L (though L_D has to be finite). We have

$$\Delta\overline{u}(y_0) = \int_{-L}^{L} dy\, G(y, y_0)\frac{\partial}{\partial y}\Delta\overline{q}(y), \tag{11.4}$$

as can be verified by subtracting $G(y, y_0)$ times (11.2) from $\Delta\overline{u}(y)$ times (11.3) and integrating with respect to y. Taking $\Delta\overline{q} = \int_{-L}^{L} dy' q_i(y')\Delta R(y', y; y_0)$, with ΔR defined by (3.17), we may integrate (11.4) by parts to give

$$\Delta\overline{u}(y_0) = -\int_{-L}^{L} dy'\, q_i(y')\hat{\eta}(y', y_0), \tag{11.5}$$

where by definition

$$\hat{\eta}(y', y_0) := \int_{-L}^{L} dy\, \Delta R(y', y; y_0) Y(y, y_0) \tag{11.6}$$

with $Y(y, y_0) := \partial G(y, y_0)/\partial y$. Now let

$$\widehat{\mathit{H}}(y', y_0) := \int_{-L}^{y'} dy''\, \hat{\eta}(y'', y_0). \tag{11.7}$$

The reasoning below (5.2)–(5.5) applies word for word to the functions $\hat{\eta}$ and $\widehat{\mathit{H}}$, after replacing the right-hand factors y and $y - y'$ in (5.2)–(5.5) by $Y(y, y_0)$ and $Y(y, y_0) - Y(y', y_0)$ respectively, y_0 being fixed throughout. It follows that $\widehat{\mathit{H}}(-L, y_0) = 0 = \widehat{\mathit{H}}(+L, y_0)$. Integrating (11.5) by parts, we therefore get a result analogous to (5.6),

$$\Delta\overline{u}(y_0) = \int_{-L}^{L} dy'\, \frac{\partial q_i(y')}{\partial y'} \widehat{\mathit{H}}(y', y_0). \tag{11.8}$$

We now use the eddy-transport-barrier assumption (11.1). The assumption says that $R(y', y; y_0)$ and $\Delta R(y', y; y_0)$ have a block diagonal structure in the $y'y$ plane, with nonvanishing values confined to two diagonal blocks meeting at $y = y' = y_0$. If $y' > y_0$, then nonvanishing contributions to $\widehat{\mathit{H}}(y', y_0)$ come from the upper right block only, and if $y' < y_0$ from the lower left only. Within each block $Y(y, y_0)$ is a monotonically increasing function of y, as will be shown next, implying that $\mathrm{sgn}[Y(y, y_0) - Y(y', y_0)] = \mathrm{sgn}(y - y')$. It will then follow that $\widehat{\mathit{H}}(y', y_0)$ is given by the right-hand side of (5.9) with $|y - y'|$ replaced by $|Y(y, y_0) - Y(y', y_0)|$, proving not only that $\widehat{\mathit{H}} \geq 0$ but also that $\Delta\overline{u}(y_0) > 0$ when $\partial q_i(y')/\partial y' > 0$, in the same way as below (5.9).

Because the reasoning below (5.9) involves a finite neighborhood in the $y'y$ plane, it is enough to prove monotonicity in the interior of each block, more specifically that $\partial Y(y, y_0)/\partial y > 0$, equivalently $\partial^2 G(y, y_0)/\partial y^2 > 0$, for $y \neq y_0$ and $y \neq \pm L$. It is here that we need the finiteness of L_D.

Consider the graph of $G(y, y_0)$ as a function of y, in each block $y < y_0$ and $y_0 < y$ separately. From (11.3) the second derivative satisfies

$$\partial^2 G(y, y_0)/\partial y^2 = L_D^{-2} G(y, y_0) \quad \text{for} \quad y \neq y_0. \tag{11.9}$$

For finite L_D the graph is therefore convex toward the y axis everywhere except at $y = y_0$ and $y = \pm L$. Because the graph goes to zero at both boundaries $y = \pm L$, it can have only the one extremum at $y = y_0$. The jump condition from (11.3),

$$\left.\frac{\partial G(y, y_0)}{\partial y}\right|_{y=y_{0-}}^{y=y_{0+}} = -1, \tag{11.10}$$

ensures that the extremum is a maximum. Therefore $G(y, y_0)$ must be positive everywhere apart from the boundaries $y = \pm L$ and therefore, from (11.9),

$$\partial^2 G(y, y_0)/\partial y^2 > 0 \quad \text{for all} \quad y \neq y_0, \pm L. \tag{11.11}$$

This completes the proof. We have established that, for both finite and infinite L, the velocity change $\Delta\overline{u}(y_0)$ in the straightened-out jet core satisfies

$$\Delta\overline{u}(y_0) > 0 \quad \text{if} \quad \partial q_i(y')/\partial y' > 0 \quad \text{for all} \quad y', \tag{11.12}$$

and vice versa, for any nontrivial R that preserves the eddy-transport barrier at the jet core.

It is not clear whether there is an alternative proof analogous to that of appendix A. The counterpart of the last term of (A.4) no longer makes a vanishing contribution to the counterpart of (A.3).

12. Beyond the present models?

It might be thought that the beta-channel results should extend to the full sphere for finite as well as for infinite L_D. However, such an extension would be far from straightforward, if only because the standard quasigeostrophic theory relies on L_D being constant. Hence for finite L_D the results are valid only to the extent that the beta channel is valid, namely, in a zone that is narrow relative to the planetary radius a and sufficiently far from the equator. A remaining challenge, therefore, is to make progress beyond the restrictions of quasigeostrophic theory and nondivergent barotropic theory, $L_D = \infty$.

Could there be an exact counterpart to the basic theorem (5.7)? The question makes sense at least for thought experiments having a zonally symmetric final as well as initial state, with both states in exact cyclostrophic balance. Then PV invertibility tells us that there is an exact counterpart to the question "what is the sign of the absolute angular-momentum change that results from generalized PV mixing?" Here "exact" indicates not only exact cyclostrophic balance but also use of the exact (Rossby–Ertel) PV.

The conservation and impermeability theorems satisfied by the exact PV (Haynes and McIntyre 1990) guarantee that the distinction between generalized mixing and unmixing is still clear. ''Particles'' of PV substance or PV charge (of either sign) can be thought of as being transported along isentropic surfaces, but never across them, even when diabatic heating is significant. Hence the upgradient transport involved in unmixing means that PV substance is transported against its isentropic gradient. Furthermore, even though the first-moment formula (3.10) fails, the total absolute angular momentum is still well defined, and exactly defined.

An exact counterpart to the basic theorem (5.7) would therefore make sense as a conjecture. However, we have so far failed to prove any such exact theorem. So the question remains open for now. The main technical obstacle appears to be the nonlinearity of the exact cyclostrophic PV inversion operator.

13. Concluding remarks

The basic theorem (5.7) proved here underlines the point that, especially in problems of jet formation and maintenance, as well as in ''beta-turbulence'' problems in general, it is advisable to consider the angular-momentum budget as well as the enstrophy and energy budgets. The theorem underlines another fundamental point as well, namely that thought experiments in which one imagines ''stirring'' the fluid to mix the PV are not well defined until one specifies what is doing the stirring. Artificial body forces will in general cause some *unmixing* of PV. So too will immersed bodies such as Welander's massless goldfish (P. B. Rhines 1971, personal communication), which produce vortex quadrupoles and are therefore capable of extending the range of PV values. Indeed massless goldfish, by definition, cannot change the absolute angular momentum. The goldfish might therefore produce profiles like those studied in DS08 and illustrated in Fig. 1c above.

Another motivation for this work was to advance our understanding of Jupiter's weather layer. An adequate representation of what we observe on the real planet will undoubtedly require a coupled model of the weather layer and the underlying convection zone. The convection zone is, in turn, bounded below by a strongly stratified transition to metallic hydrogen, as the pressure increases and the proportion of ionized hydrogen atoms to neutral atoms and H_2 molecules builds up with temperature. It is likely that Richardson numbers in the transition zone are enormous. So it may well be that one can treat the transition zone as a rigid but perfectly slippery boundary, whose only function is to supply heat from below.

Our current aim is less ambitious, namely to isolate one aspect of the coupling between the top of the convection zone and the overlying weather layer by making the simplifying assumption that the main effect of the convection zone is to exert the fluctuating form stress required to catalyze PV mixing and jet formation. For instance, the form stress can be exerted via an artificial ''heaving topography'' $\tilde{b}(x, y, t)$ acting as the forcing function on a shallow-water layer, in place of the usual artificial body forces. Arguably, the addition of such quasi-topographic forcing might improve the realism of simulations like that of Showman (2007). Showman also avoids using artificial body forces, but assumes that the sole effect of the convection zone is to produce small-scale mass injections into the weather layer, like thunderstorm anvils.

A further question is whether, with a more natural and realistic forcing, we can reach a statistically steady state without having to invoke large-scale Rayleigh friction or hypodiffusion, both of which are hardly natural assumptions for a planet with no nearby solid surface.

These questions are as yet unanswered but we hope to make progress on them soon, through numerical experiments based on sophisticated numerical codes that as far as possible respect the angular-momentum principle.

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APPENDIX A

An Alternative Proof

The connection to potential energy noted in section 6 suggests an alternative proof of the basic theorem via a mathematical route quite different from that of section 5. It is motivated by positive-definite exact formulas for potential-energy changes that are already known for purely advective rearrangements of buoyancy (e.g., Holliday and McIntyre 1981; Andrews 1981; Molemaker and McWilliams 2010; Roullet and Klein 2009). These exact formulas are now recognized as cases of the energy–Casimir and momentum–Casimir formulas arising in Hamiltonian models of disturbances to nontrivial initial or background states (e.g., Arnol'd 1965; Shepherd 1993). The resulting proof of (5.7) can be seen as a nontrivial generalization of the Hamiltonian theory, made possible by the *R*-function formalism.

For a purely advective rearrangement, the Hamiltonian formulas apply. In the shallow-water case, for instance, we have

$$\Delta M = -\rho_0 H \int_{-L}^{L} dy\, A(y, \breve{\eta}), \qquad (A.1)$$

where $\breve{\eta}$ is the latitudinal displacement of a fluid element expressed as a function of its final latitude y rather than its initial latitude y', so that $\breve{\eta}(y) = \eta(y') = y - y'$, and where the function A is defined by

$$A(y, \breve{\eta}) := \int_0^{\breve{\eta}} d\breve{\eta}^\dagger \frac{\partial q_i(y - \breve{\eta}^\dagger)}{\partial y} \breve{\eta}^\dagger. \qquad (A.2)$$

It is only because of the invertible mapping between the initial latitude y' and final latitude y of a given fluid element, in the purely advective case, that we can write the displacement of that element as a function either of y' or of y.

We now show that the R-function formalism allows us to rederive (A.1) together with its generalization beyond the Hamiltonian framework, as a single expression

$$\Delta M = -\rho_0 H \int_{-L}^{L} dy \int_{-L}^{L} dy' R(y', y) A(y, y - y'). \qquad (A.3)$$

First, we see by inspection that in the purely advective case, for which $R(y', y) = \delta[y - y' - \breve{\eta}(y)]$, the expression (A.3) does reproduce (A.1). Second, to see that (A.3) is correct for a general R function, we rewrite (A.2) by substituting $y^\dagger := y - \breve{\eta}^\dagger$ and integrating by parts to obtain

$$A(y, y - y') = -q_i(y')(y - y') - \int_y^{y'} dy^\dagger\, q_i(y^\dagger). \qquad (A.4)$$

Now the last term of (A.4) contributes nothing to (A.3). This is because it has the functional form $a(y) - a(y')$. In virtue of the integral constraints (2.7) and (2.8), the contribution to (A.3) is $\int_{-L}^{L}\int_{-L}^{L} dy\, dy'\, R(y', y)[a(y) - a(y')] = \int_{-L}^{L} a(y)\, dy - \int_{-L}^{L} a(y')\, dy' = 0$, for any function $a(\cdot)$.

The definition (5.2) of the average displacement $\eta(y')$ can be rewritten using (2.7) and (3.17) as

$$\begin{aligned}\eta(y') &= \int_{-L}^{L} dy\, \Delta R(y', y)(y - y') \\ &= \int_{-L}^{L} dy\, R(y', y)(y - y'). \qquad (A.5)\end{aligned}$$

Hence, by substituting the first term of (A.4) into (A.3), then using (A.5) to rewrite the result in terms of $\eta(y')$, we see that (A.3) is equivalent to the original expression (5.1) for ΔM. The basic theorem (5.7) now follows, because (A.2) shows that the function A is positive definite whenever $\partial q_i/\partial y$ is positive, and negative definite whenever $\partial q_i/\partial y$ is negative.

APPENDIX B

Monotonizing Q

To see how to obtain monotonically increasing $q_2[y'; q_1(\cdot)]$ from the general $q_1(x', y')$ by advective rearrangement, one may proceed as follows.

The function describing the monotonic PV distribution $q = q_2[y; q_1(\cdot)]$ will have an inverse function $y = y_2[q; q_1(\cdot)]$. For a given q value, all the fluid with $q_1 > q$ will, after rearrangement, lie between $y = y_2[q; q_1(\cdot)]$ and the northern boundary $y = L$. Hence we may define

$$y_2[q; q_1(\cdot)] := L - \int_{-L}^{L} dy\, \overline{\mathcal{H}\{q_1(x, y) - q\}}, \quad (B.1)$$

where $\mathcal{H}$ is the Heaviside step function and the overbar again denotes averaging in x. The redistribution function representing the advective rearrangement from $q_1(x', y')$ to $q_2[y'; q_1(\cdot)]$ is

$$s[x', y', x, y; q_1(\cdot)] = \frac{1}{\int dx'} \delta\{y - y_2[q_1(x', y'); q_1(\cdot)]\}, \qquad (B.2)$$

where δ is the Dirac delta function. Equation (B.2) can be verified by substituting this s into (9.1).

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